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B. Sc. (Honrs) Part 3 paper 5

Subject: Mathematics

Title/Heading of topic: Definition and existence of  
Riemann integral of bounded function

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## : Riemann Integration

Integration is concerned with the problem of finding the area of a region under a curve.

Let us start with a simple problem : *Find the area  $A$  of the region enclosed by a circle of radius  $r$ .* For an arbitrary  $n$ , consider the  $n$  equal inscribed and superscribed triangles as shown in Figure 1.

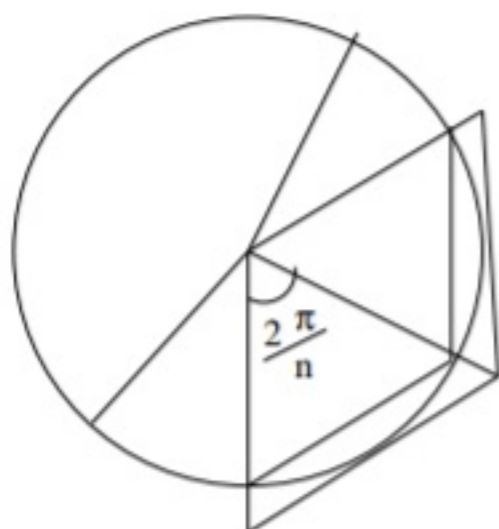


Figure 1

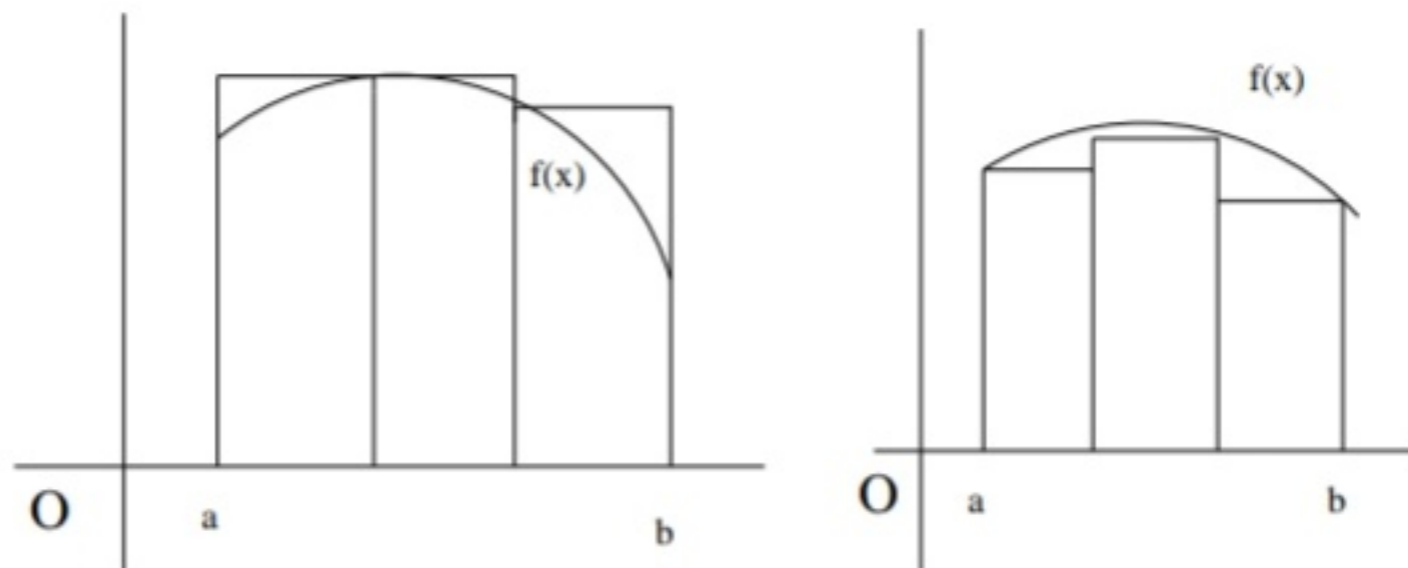


Figure 2

Since  $A$  is between the total areas of the inscribed and superscribed triangles, we have

$$nr^2 \sin(\pi/n) \cos(\pi/n) \leq A \leq nr^2 \tan(\pi/n).$$

By sandwich theorem,  $A = \pi r^2$ . **We will use this idea to define and evaluate the area of the region under a graph of a function.**

Suppose  $f$  is a non-negative function defined on the interval  $[a, b]$ . We first subdivide the interval into a finite number of subintervals. Then we squeeze the area of the region under the graph of  $f$  between the areas of the inscribed and superscribed rectangles constructed over the subintervals as shown in Figure 2. **If the total areas of the inscribed and superscribed rectangles converge to the same limit as we make the partition of  $[a, b]$  finer and finer then the area of the region under the graph of  $f$  can be defined as this limit and  $f$  is said to be integrable.**

*Let us define whatever has been explained above formally.*

### The Riemann Integral

Let  $[a, b]$  be a given interval. A *partition*  $P$  of  $[a, b]$  is a finite set of points  $x_0, x_1, x_2, \dots, x_n$  such that  $a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b$ . We write  $P = \{x_0, x_1, x_2, \dots, x_n\}$ .

If  $P$  is a partition of  $[a, b]$  we write  $\Delta x_i = x_i - x_{i-1}$  for  $1 \leq i \leq n$ . Let  $f$  be a bounded real valued function on  $[a, b]$ . For a partition  $P$  of  $[a, b]$ , we define

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \text{ and } m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i \text{ and } L(P, f) = \sum_{i=1}^n m_i \Delta x_i.$$

The numbers  $U(P, f)$  and  $L(P, f)$  are called *upper and lower Riemann sums* for the partition  $P$  (see Figure 2).

Since  $f$  is bounded, there exist real numbers  $m$  and  $M$  such that  $m \leq f(x) \leq M$ , for all  $x \in [a, b]$ . Thus for every partition  $P$ ,

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a).$$

We define

$$\overline{\int_a^b} f dx = \inf U(P, f) \quad (1)$$

and

$$\underline{\int_a^b} f dx = \sup L(P, f). \quad (2)$$

(1) and (2) are called *upper and lower Riemann integrals* of  $f$  over  $[a, b]$  respectively.

If the upper and lower integrals are equal, we say that  $f$  is *Riemann integrable* or *integrable*. In this case the common value of (1) and (2) is called the Riemann integral of  $f$  and is denoted by  $\int_a^b f dx$  or  $\int_a^b f(x) dx$ .

**Examples :** 1. Consider the function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by

$$f\left(\frac{1}{2}\right) = 1 \text{ and } f(x) = 0 \text{ for all } x \in [0, 1] \setminus \left\{\frac{1}{2}\right\}.$$

Then  $f$  is integrable. We show this using the definition as follows. For any partition  $P$  of  $[0, 1]$ ,  $L(P, f)$  is always 0 and hence the lower integral is 0. Let us evaluate the upper integral. Let  $P = \{x_1, x_2, \dots, x_n\}$  be any partition of  $[0, 1]$  and  $\frac{1}{2} \in [x_i, x_{i+1}]$  for some  $i$ . Then  $U(P, f) \leq 2 \max \Delta x_j$ . Since we can always choose a partition  $P$  such that  $\max \Delta x_j$  is as small as possible, the upper integral, which is the infimum of  $U(P, f)$ 's, is 0. Hence,  $f$  is integrable and  $\int_0^1 f(x) dx = 0$ .

2. Not every bounded function is integrable. For example the function

$$f(x) = 1 \text{ if } x \text{ is rational and } 0 \text{ otherwise}$$

is not integrable over any interval  $[a, b]$  (Check this).

In general, determining whether a bounded function on  $[a, b]$  is integrable, using the definition, is difficult. For the purpose of checking the integrability, we give a criterion for integrability, called Riemann criterion, which is analogous to the Cauchy criterion for the convergence of a sequence.

Let us define some concepts and results before presenting the criterion. Throughout, we will assume that  $f$  is a bounded real function on  $[a, b]$ .

**Definition:** A partition  $P_2$  of  $[a, b]$  is said to be finer than a partition  $P_1$  if  $P_2 \supset P_1$ . In this case we say that  $P_2$  is a *refinement* of  $P_1$ . Given two partition  $P_1$  and  $P_2$ , the partition  $P_1 \cup P_2 = P$  is called their common refinement.

The following theorem illustrates that refining partition increases lower terms and decreases upper terms.

**Theorem 1 :** Let  $P_2$  be a refinement of  $P_1$  then  $L(P_1, f) \leq L(P_2, f)$  and  $U(P_2, f) \leq U(P_1, f)$ .

**Proof (\*):** First we assume that  $P_2$  contains just one more point than  $P_1$ . Let this extra point be  $x^*$ . Suppose  $x_{i-1} < x^* < x_i$ , where  $x_{i-1}$  and  $x_i$  are consecutive points of  $P_1$ . Let

$$\begin{aligned} w_1 &= \inf\{f(x) : x_{i-1} \leq x \leq x^*\} \text{ and} \\ w_2 &= \inf\{f(x) : x^* \leq x \leq x_i.\} \end{aligned}$$

Then  $w_1 \geq m_i$  and  $w_2 \geq m_i$  where  $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ . Then

$$L(P_2, f) - L(P_1, f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1}) \geq 0.$$

If  $P_2$  contains  $k$  more points then we repeat this process  $k$ -times. The other inequality is analogously proved. (Prove it).  $\square$

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

**Corollary 2 :**  $\int_a^b f dx \geq \underline{\int_a^b} f dx$ .

**Proof (\*) :** Let  $P_1, P_2$  be two partitions and let  $P$  be their common refinement. Then

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f).$$

Thus for any two partitions  $P_1$  and  $P_2$ , we have  $L(P_1, f) \leq U(P_2, f)$ .

Fix  $P_2$  and take sup over all  $P_1$ . Then  $\underline{\int_a^b} f dx \leq U(P_2, f)$ . Now take inf over all  $P_2$  to get the desired result.  $\square$

In the following result we present the Riemann criterion (a necessary and sufficient condition for the existence of the integral of a bounded function).

**Theorem 3 : (Riemann's criterion for integrability):**  $f$  is integrable on  $[a, b] \Leftrightarrow$  for every  $\epsilon > 0$  there exists a partition  $P$  such that

$$U(P, f) - L(P, f) < \epsilon. \quad (3)$$

**Proof (\*) :** For any  $P$ , we have

$$L(P, f) \leq \underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx \leq U(P, f).$$

Therefore (3) implies that

$$\overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \epsilon, \quad \forall \epsilon > 0.$$

Hence  $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$  i.e.  $f$  is integrable. Conversely, suppose  $f$  is integrable and  $\epsilon > 0$ . Then there exist partitions  $P_1$  and  $P_2$  such that

$$U(P_2, f) - \underline{\int_a^b} f dx < \epsilon/2 \quad \text{and} \quad \overline{\int_a^b} f dx - L(P_1, f) < \epsilon/2$$

Let  $P$  be the common refinement of  $P_1$  and  $P_2$ . Then  $U(P, f) - L(P, f) < \epsilon$ .  $\square$

The proof of the following corollary is immediate from the previous theorem.

**Corollary 3 :** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Suppose  $(P_n)$  is a sequence of partitions of  $[a, b]$  such that  $U(P_n, f) - L(P_n, f) \rightarrow 0$ , then  $f$  is integrable.

**Problem :** Let  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$  Show that  $f$  is integrable and find  $\int_0^1 f(x) dx$ .

**Solution :** We will use the Riemann criterion to show that  $f$  is integrable on  $[0, 1]$ . Let  $\epsilon > 0$  be given. We will choose a partition  $P$  such that  $U(P, f) - L(P, f) < \epsilon$ . Since  $1/n \rightarrow 0$ , there exists  $N$  such that  $1/n \in [0, \epsilon]$  for all  $n > N$ . So only finite number of  $\frac{1}{n}$ 's lie in the interval  $[\epsilon, 1]$ . Cover these finite number of  $\frac{1}{n}$ 's by the intervals  $[x_1, x_2], [x_3, x_4], \dots, [x_{m-1}, x_m]$  such that  $x_i \in [\epsilon, 1]$  for all

$i = 1, 2, \dots, m$  and the sum of the length of these  $m$  intervals is less than  $\varepsilon$ . Consider the partition  $P = \{0, \varepsilon, x_1, x_2, \dots, x_m\}$ . It is clear that  $U(P, f) - L(P, f) < 2\varepsilon$ . Hence by the Riemann criterion the function is integrable. Since the lower integral is 0 and the function is integrable,  $\int_0^1 f(x)dx = 0$ .

We will apply the Riemann criterion for integrability to prove the following two existence theorems.

**Theorem 4:** *If  $f$  is continuous on  $[a, b]$  then  $f$  is integrable.*

**Proof :** Let  $\varepsilon > 0$ . Since  $f$  is uniformly continuous, choose  $\delta > 0$  such that  $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$  for  $s, t \in [a, b]$ .

Let  $P$  be a partition of  $[a, b]$  such that  $\Delta x_i < \delta \forall i = 1, 2, \dots, n$ . Then

$$M_i - m_i \leq \varepsilon \quad \forall i = 1, 2, \dots, n.$$

Hence

$$U(P, f) - L(P, f) = \sum_1^n (M_i - m_i) \Delta x_i \leq \varepsilon(b - a).$$

This implies that  $f$  is integrable. □

**Theorem 5:** *If  $f$  is a monotone function then  $f$  is integrable.*

**Proof :** Suppose  $f$  is monotonically increasing (the proof is similar in the other case.) Choose a partition  $P$  such that  $\Delta x_i = \frac{b-a}{n}$ . Then  $M_i = f(x_i)$  and  $m_i = f(x_{i-1})$ . Therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \frac{b-a}{n} \sum_1^n [f(x_i) - f(x_{i-1})] \\ &= \frac{b-a}{n} [f(b) - f(a)] \\ &< \varepsilon \quad \text{for large } n. \end{aligned}$$

Hence  $f$  is integrable. □

In the following problem we will see that limit and integral cannot be interchanged.

**Problem :** Let  $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \leq y < 1 \\ 0 & \text{if } y = 1 \end{cases}$ . Then prove that  $\lim_{n \rightarrow \infty} \int_0^1 g_n(y)dy = \frac{1}{2}$  whereas

$$\int_0^1 \lim_{n \rightarrow \infty} g_n(y)dy = 0.$$

**Solution :** From the ratio test for sequences we can show that  $\lim_{n \rightarrow \infty} \frac{ny^{n-1}}{1+y} = 0$ , for each  $0 < y < 1$ .

Therefore  $\int_0^1 \lim_{n \rightarrow \infty} g_n(y)dy = 0$ .

For the other part, use integration by parts to see that  $\int_0^1 \frac{ny^{n-1}}{1+y} dy = \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2} dy$ . Note that

$$\int_0^1 \frac{y^n}{(1+y)^2} dy \leq \int_0^1 y^n = \frac{1}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty. \text{ Therefore, } \lim_{n \rightarrow \infty} \int_0^1 g_n(y)dy = \frac{1}{2}.$$